WEAK AND STRONG PARETO OPTIMALITY

DEFN: FOR GIVEN PREFERENCES $(\succsim_i)_{i \in N}$ OVER A SET X OF ALTERNATIVES, AN ALTERNATIVE $\tilde{x}$ IS A WEAK PARETO IMPROVEMENT UPON ALTERNATIVE x IF

$$(a) \quad \tilde{x} \succsim_i x$$

HOLDS FOR ALL $i \in N$, AND

$$(b) \quad \tilde{x} \succ_i x$$

HOLDS FOR SOME $i \in N$. THE ALTERNATIVE $\tilde{x}$ IS A STRONG PARETO IMPROVEMENT UPON x IF (b) HOLDS FOR ALL $i \in N$. AN ALTERNATIVE $\hat{x}$ IS A STRONG PARETO OPTIMUM (WITH RESPECT TO X) IF NO ALTERNATIVE IN X IS A WEAK PARETO IMPROVEMENT ON $\hat{x}$; AND $\hat{x}$ IS A WEAK PARETO OPTIMUM IF NO ALTERNATIVE IN X IS A STRONG PARETO IMPROVEMENT ON $\hat{x}$.

REMARK: IT IS A TRIVIAL CONSEQUENCE OF THE DEFINITIONS THAT A STRONG PARETO IMPROVEMENT IS ALSO A WEAK PARETO IMPROVEMENT, AND THUS THAT A STRONG PARETO OPTIMUM IS ALSO A WEAK PARETO OPTIMUM.

<u>THEOREM:</u> IF EACH PREFERENCE $\succsim_i$ IS CONTINUOUS, SELFISH, AND STRICTLY INCREASING ON A SET X OF ALLOCATIONS IN $\mathbb{R}^{n\ell}_+$, THEN WHENEVER A WEAK PARETO IMPROVEMENT ON AN ALLOCATION $\hat{x}$ EXISTS, A STRONG PARETO IMPROVEMENT EXISTS AS WELL.

<u>PROOF:</u>

LET $\hat{x}$ BE AN ALLOCATION AND LET $\tilde{x}$ BE A WEAK PARETO IMPROVEMENT ON $\hat{x}$ — i.e., $\sum_N \tilde{x}^i \leqq \mathring{x}$, AND (a) IN THE DEFINITION ABOVE IS SATISFIED $\forall i \in N$, AND (b) IS SATISFIED (WLOG) FOR $i=1$. SINCE $u^1(\tilde{x}^1) > u^1(\hat{x}^1)$, SOME COMPONENT OF $\tilde{x}^1$, SAY $\tilde{x}^1_k$, IS STRICTLY LARGER THAN THE MINIMUM VALUE OF x^1_k ADMISSIBLE FOR $i=1$. LET e_k BE THE k^{TH} UNIT VECTOR IN $\mathbb{R}^\ell$; IF $t>0$ IS SMALL ENOUGH, THEN (BECAUSE u^1 IS CONTINUOUS) WE WILL HAVE $u^1(\tilde{x}^1 - te_k) > u^1(\hat{x}^1)$. DEFINE AN ALLOCATION $\bar{x}$ AS FOLLOWS:

$$\bar{x}^i = \begin{cases} \tilde{x}^i - te_k, & \text{IF } i=1 \\ \tilde{x}^i + \frac{t}{n-1} e_k, & \text{IF } i \neq 1. \end{cases}$$

CLEARLY, $u^i(\bar{x}^i) > u^i(\hat{x}^i)$ FOR EACH $i \neq 1$, BECAUSE EACH u^i IS STRICTLY INCREASING. MOREOVER,

$$\sum_{i \in N} \bar{x}^i = \tilde{x}^1 - te_k + \sum_{i \neq 1}\left(\tilde{x}^i + \frac{t}{n-1} e_k\right) = \sum_{i \in N} \tilde{x}^i \leqq \mathring{x},$$

SO WE HAVE $(\bar{x}^i)_1^n$ FEASIBLE. HENCE, $(\bar{x}^i)_1^n$ IS A STRONG PARETO IMPROVEMENT UPON $(\hat{x}^i)_1^n$. ||

<u>COROLLARY:</u> IF EACH PREFERENCE IS CONTINUOUS, SELFISH, AND STRICTLY INCREASING, THEN A WEAK PARETO OPTIMUM IS ALSO A STRONG PARETO OPTIMUM — i.e., THEN THE TWO NOTIONS OF PARETO EFFICIENCY ARE EQUIVALENT.